PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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J. R. Holton

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# Wave propagation and transport in the middle atmosphere

By J. R. Holton
*Department of Atmospheric Sciences, University of Washington, Seattle, Washington* 98195 *U.S.A.*

The dynamics of wave propagation and wave transport are reviewed for vertically propagating, forced, planetary scale waves in the middle atmosphere. Such waves can be divided into two major classes: extratropical planetary waves and equatorial waves. The most important waves of the former class are quasi-stationary Rossby modes of zonal wavenumbers 1 and 2 (1 or 2 waves around a latitude circle), which propagate vertically only during the winter season when the mean winds are westerly. These modes transport heat and ozone towards the poles, thus maintaining the mean temperature above its radiative equilibrium value in high latitudes and producing the high latitude ozone maximum. It is shown that these wave transport processes depend on wave transience and wave damping. The precise form of this dependency is illustrated for transport of a strongly stratified tracer by small amplitude planetary waves.

The observed equatorial wave modes are of two types: an eastward propagating Kelvin mode and a westward propagating mixed Rossby–gravity mode. These modes are thermally damped in the stratosphere where they interact with the mean flow to produce eastward and westward accelerations, respectively. It is shown that in the absence of mechanical dissipation this wave–mean flow interaction is caused by the vertical divergence of a wave 'radiation stress'. This wave–mean flow interaction process is responsible for producing the well known equatorial quasi-biennial oscillation.

## 1. Introduction

Planetary scale wave motions are essential components of the general circulation in the middle atmosphere. Much of the observed temporal and spatial variability in winds, temperatures, and trace species concentrations in this region is due directly or indirectly to wave motions. Atmospheric waves play major roles in maintaining the zonal mean momentum and temperature budgets as well as the ozone budget. Such waves can be classified according to their horizontal structures, their vertical structures, their sources of excitation, and their modes of interaction with the mean flow.

This classification scheme allows planetary scale waves to be categorized on the basis of the following dualities: (1) extratropical modes against equatorially trapped modes, (2) free modes against forced modes, (3) external modes against internal modes, and (4) modes that interact with the mean flow through wave transience against modes that interact through wave dissipation. Of these various possible wave types the waves of primary importance for middle atmosphere dynamics are forced internal modes which are excited by various processes in the troposphere and propagate vertically into the middle atmosphere. The most significant forced vertically propagating extratropical modes are the quasi-stationary Rossby waves, while the most significant forced vertically propagating equatorial modes are the Kelvin wave and the mixed Rossby–gravity wave. Both the extratropical modes and the equatorial modes are

† Contribution no. 520, Department of Atmospheric Sciences, University of Washington, Seattle.

capable of generating mean flow changes through the process of wave–mean flow interaction. The quasi-stationary Rossby waves interact with the mean flow primarily through the mechanism of wave transience (local changes in wave amplitude) while the equatorial waves interact with the mean flow primarily through the mechanism of wave dissipation (thermal and/or mechanical damping). In both cases the waves transfer momentum vertically from the tropospheric source region into the middle atmosphere. In the following sections these wave–mean flow interaction processes are discussed for both extratropical waves and equatorial waves.

## 2. Extratropical waves

The zonally asymmetric component of the circulation in the winter hemisphere middle atmosphere is dominated by quasi-stationary Rossby waves of zonal wavenumbers 1 and 2. These waves are merely the upward extensions of tropospheric planetary waves generated by topographic forcing and by land–sea diabatic heating contrasts. Although both types of forcing are strongest during the winter season, the confinement of these waves to the winter hemisphere in the middle atmosphere is not due to seasonal changes in the forcing, but rather to the strong dependence of the vertical wave transmission on the mean zonal wind profile. Likewise, oscillations in wave amplitude and phase may occur not only as a result of oscillations in the forcing, but also in response to changes in the transmission characteristics of the middle atmosphere due to mean zonal wind changes. In addition, travelling free modes may alternately constructively and destructively interfere with the stationary forced modes to produce wave amplitude variability as has been discussed, for example, by Leovy & Webster (1976).

An explanation of the observed confinement of quasi-stationary planetary waves to the winter hemisphere in the middle atmosphere was first provided by Charney & Drazin (1961). Although their analysis was carried out using spherical geometry, the essential physics may be demonstrated using a mid-latitude '$\beta$-plane' in which the spherical geometry is replaced by Cartesian coordinates with $x$ directed eastward and $y$ northward, but the dynamical effects of the variation of the Corolis parameter with latitude are retained by setting $\mathrm{d}f/\mathrm{d}y = \beta =$ constant. The quasi-geostrophic potential vorticity equation can then be written (Holton 1975)

$$\mathrm{d}q/\mathrm{d}t = S, \tag{1}$$

where
$$q \equiv f + \nabla^2\psi + \frac{f^2}{N^2}\rho_0^{-1}\frac{\partial}{\partial z}\left(\rho_0\frac{\partial\psi}{\partial z}\right),$$

with
$$\nabla^2 \equiv \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y^2}; \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u_\psi\frac{\partial}{\partial x} + v_\psi\frac{\partial}{\partial y};$$

and $S$ designates all sources and sinks of potential vorticity. Here, $\psi$ is a geostrophic streamfunction so that $u_\psi = -\psi_y$, $v_\psi = \psi_x$; $f$ is the Coriolis parameter; $N$ is the buoyancy frequency; $\rho_0 \equiv \rho_0(z)$ is the basic state density; $z \equiv -H\ln(p/p_s)$ is a log-pressure vertical coordinate; $p$ is the local pressure; $p_s$ is a standard reference pressure; $H$ is a constant scale height.

Letting $\psi = \bar{\psi} + \psi'$ and $q = \bar{q} + q'$, where the overbars designate zonal means and primes designate deviations, the zonal mean of (1) can be written as

$$\partial\bar{q}/\partial t = -\partial(\overline{q'\psi'_x})/\partial y + \bar{S}. \tag{2}$$

Assuming that wave perturbations are $O(a)$, where $a$ designates a small amplitude, the perturbation form of (1) correct to $O(a)$ is the linear wave equation

$$\frac{\partial q'}{\partial t}+\bar{u}\,\frac{\partial q'}{\partial x}+\frac{\partial \psi'}{\partial x}\,\frac{\partial \bar{q}}{\partial y} = S', \tag{3}$$

where
$$q' = \nabla^2\psi'+\frac{1}{\rho_0}\,\frac{\partial}{\partial z}\left(\frac{\rho_0 f^2}{N^2}\,\frac{\partial \psi'}{\partial z}\right)$$

and
$$\frac{\partial \bar{q}}{\partial y} = \beta-\frac{\partial^2 \bar{u}}{\partial y^2}-\frac{1}{\rho_0}\,\frac{\partial}{\partial z}\left(\frac{\rho_0 f^2}{N^2}\,\frac{\partial \bar{u}}{\partial z}\right).$$

*(a) The Charney–Drazin criterion*

If it is assumed that $\bar{u} \equiv \bar{u}(z)$ the linear wave equation (3) has separable solutions of the form

$$\psi' = \Psi(z)\,\rho_0^{-\frac{1}{2}}\exp\left[\mathrm{i}(kx+ly-kct)\right]. \tag{4}$$

Substitution from (4) into (3) and neglecting perturbation sources and sinks by setting $S' = 0$ yields the vertical structure equation

$$\Psi_{zz}+n^2\Psi = 0, \tag{5}$$

where
$$n^2 \equiv \frac{N^2}{f^2}\left[\frac{\bar{q}_y}{(\bar{u}-c)}-(k^2+l^2)\right]-\frac{1}{4H^2}.$$

For $n^2 > 0$ (5) has solutions in the form of internal (vertically propagating) waves while for $n^2 \leqslant 0$ the solutions are external (vertically trapped) waves. For stationary waves ($c = 0$) vertical propagation is thus possible only for mean zonal winds satisfying

$$0 < \bar{u} < U_c, \tag{6}$$

where $U_c \equiv \bar{q}_y(k^2+l^2+f^2/4N^2H^2)^{-1}$ is the *Rossby critical velocity*. Thus, vertical propagation for stationary waves can occur only if the mean winds are westerly but less than a critical velocity $U_c$. This criterion, first obtained by Charney & Drazin (1961), is of course based on a highly simplified model of the mean zonal wind. Simmons (1974) and others have shown that it is important to consider the meridional variation of $\bar{u}$ and that for realistic mean wind conditions vertical propagation is possible for somewhat larger mean winds than indicated by the Charney–Drazin criterion. However, qualitatively, equation (6) provides a valid guide for propagation of stationary Rossby waves. In particular, the strong dependence of $U_c$ on wavelength has been confirmed by many models and is undoubtedly the reason for the observed dominance of quasi-stationary zonal wavenumbers 1 and 2 in the winter season.

The behaviour of stationary Rossby waves in the vicinity of a *critical level* where $\bar{u} = 0$ (and hence $n^2 \rightarrow \infty$) has been explored in a number of theoretical studies. Dickinson (1970), on the basis of linear theory, found that Rossby waves propagating meridionally towards the $\bar{u} = 0$ line near the equator would be absorbed near the critical line and hence produce easterly accelerations of the mean flow. More recent calculations by Béland (1976) indicate that under most conditions the Rossby wave critical level is a nonlinear phenomenon and that the waves are reflected rather than absorbed. Observations (Van Loon *et al.* 1973) do, however, indicate that in the Northern Hemisphere winter, zonal wavenumber 1 has phase lines that tilt with latitude in the vicinity of the equatorial critical level, thus suggesting absorption at the critical line. The Rossby wave critical level problem has, therefore, not yet been completely resolved.

The question is an important one, however, because Tung & Lindzen (1978) have shown that if the critical line behaviour is nonlinear, then for certain distributions of $\bar{u}(y, z)$ stationary Rossby waves might become resonant and amplify anomalously to produce sudden stratospheric warmings as well as changes in the planetary wave structure in the troposphere.

### (b) *Wave–mean flow interaction*

The dynamical processes through which quasi-stationary Rossby waves influence the mean circulation in the stratosphere and mesosphere can be elucidated through consideration of potential vorticity conservation. According to the zonal mean quasi-geostrophic potential vorticity equation (2), in the absence of mean sources or sinks ($\bar{S} = 0$), $\bar{q}$ can be changed only if there is a meridional flux of eddy potential vorticity, $\overline{q'v'_\psi}$. This eddy flux can be expressed in terms of wave transience and wave dissipation for the case of linear perturbations by multiplying (3) by $q'$ and averaging zonally to obtain

$$\overline{q'v'_\psi} = [\overline{q'S'} - \partial(\tfrac{1}{2}\overline{q'^2})/\partial t]/\bar{q}_y. \tag{7}$$

Thus, provided that $\bar{S} = 0$,

$$\frac{\partial \bar{q}}{\partial t} = \left[-\frac{\partial}{\partial y}(\overline{q'S'}) + \frac{\partial^2}{\partial t\,\partial y}(\tfrac{1}{2}\overline{q'^2})\right]\left(\frac{\partial \bar{q}}{\partial y}\right)^{-1}. \tag{8}$$

It is then immediately obvious that, in the absence of wave transience, ($\partial\overline{q'^2}/\partial t = 0$) and wave dissipation ($S' = 0$) $\bar{q}$ is constant in time so that there is no mean flow forcing by the waves. This result is the famous Charney–Drazin non-interaction theorem (Charney & Drazin 1961), which has recently been greatly generalized by Andrews & McIntyre (1976*a*, 1978). From (7) it is clear that waves which are increasing in amplitude or undergoing damping ($\overline{q'S'} < 0$) will produce potential vorticity fluxes *down* the gradient of mean potential vorticity. Decaying waves, on the other hand, will produce countergradient fluxes of potential vorticity. Thus, the eddy flux of potential vorticity cannot in general be parametrized as a simple eddy diffusion process.

The potential vorticity flux due to wave transience is of special significance for the general circulation of the middle atmosphere. A number of modelling studies have indicated that wave transience plays an essential role in the development of sudden stratospheric warmings (see, for example, Matsuno 1971; Holton 1976; Holton & Mass 1976). The primacy of the wave transience contribution to mean flow forcing in the sudden warming situation has been specifically demonstrated by Holton & Dunkerton (1978), who explicitly computed the two terms on the right in equation (8) for a $\beta$-plane channel model of forced wave–mean flow interaction. Because of the large vertical scale and consequent small temperature perturbations in the observed quasi-stationary Rossby waves of the winter hemisphere, thermal damping processes are not very important on the rapid (*ca.* 7 days) timescale of the sudden warmings.

### (c) *Wave transport of vertically stratified tracers*

Quasi-stationary Rossby waves serve not only to transport potential vorticity, and hence to drive changes in the mean flow, but also transport trace constituents such as ozone. It turns out that the *net* wave transport again depends on wave transience and dissipation. A particularly simple and instructive analysis of the wave transport problem can be carried out for the special, but important, case of a strongly vertically stratified tracer. This analysis should be approximately valid for the ozone mixing ratio below its peak near 30 km, and is also applicable to the transport of potential temperature.


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As in the previous subsection the analysis is based on quasi-geostrophic flow on the mid-latitude $\beta$-plane. Thus, the horizontal flow is quasi-non-divergent;

$$\partial u/\partial x = \partial v/\partial y \approx O(\epsilon), \tag{9}$$

where $\epsilon \ll 1$ is the Rossby number. It is assumed that the tracer follows a conservation law

$$\mathrm{d}X/\mathrm{d}t = S, \tag{10}$$

where $X$ is the mixing ratio (alternatively, $X$ could denote the potential temperature) and $S$ designates the sum of all sources and sinks.

For strong vertical stratification it is convenient to define a basic state concentration dependent only on height such that

$$X(x, y, z, t) = X_0(z) + \chi(x, y, z, t). \tag{11}$$

In the following analysis it is assumed that

$$\frac{\partial \chi}{\partial y} \Big/ \frac{\mathrm{d}X_0}{\mathrm{d}z} \approx O(\epsilon\delta),$$

where $\delta \equiv D/L$ is the aspect ratio, that is, the ratio of the vertical scale $D$ to the meridional scale $L$. With the aid of (9) and (11) the conservation equation (10) can then be written in flux form correct to $O(\epsilon)$ as follows:

$$\frac{\partial \chi}{\partial t} + \frac{\partial}{\partial x}(\chi u) + \frac{\partial}{\partial y}(\chi v) + w\frac{\mathrm{d}X_0}{\mathrm{d}z} = S. \tag{12}$$

The concentration is now separated into zonal mean and eddy components by letting $\chi = \bar{\chi} + \chi'$, where $\chi'$ is assumed to be an $O(a)$ deviation associated with the planetary wave disturbances. Taking the zonal mean of (10) then yields

$$\frac{\partial \bar{\chi}}{\partial t} = -\frac{\partial}{\partial y}(\overline{\chi' v'}) - \bar{w}\frac{\mathrm{d}X_0}{\mathrm{d}z} + \bar{S}, \tag{13}$$

while the perturbation equation correct to $O(a)$ becomes

$$\frac{\partial \chi'}{\partial t} + \bar{u}\frac{\partial \chi'}{\partial x} + v'\frac{\partial \bar{\chi}}{\partial y} + w'\frac{\mathrm{d}X_0}{\mathrm{d}z} = S'. \tag{14}$$

According to (13) the zonal mean tracer concentration for a strongly stratified tracer is determined by a balance among three processes: (1) horizontal eddy flux convergence, (2) vertical advection of the basic state, and (3) zonal mean sources and sinks. (If $\chi$ designates potential temperature these terms represent the eddy heat flux convergence, the adiabatic heating, and diabatic heating, respectively.) It is important to note that the transport effect of the waves is not limited to the eddy flux term because the Eulerian mean vertical motion $\bar{w}$ is itself partly driven by the waves. In fact, if $\bar{S} = 0$ and wave transience and dissipation vanish, the advection term $\bar{w}\mathrm{d}X_0/\mathrm{d}z$ exactly balances the horizontal eddy flux convergence so that there is no net transport. Thus, wave transience and dissipation play the same important roles for transport of conservative tracers as they play in the momentum and heat budgets.

In order to examine explicitly the relation of transport to transience and dissipation it is convenient to follow Andrews & McIntyre (1976a) by introducing particle displacement fields $\xi'(x, y, z, t)$, $\eta'(x, y, z, t)$, $\zeta'(x, y, z, t)$, such that

$$D_t\xi' = u' + \eta'\,\partial\bar{u}/\partial y; \quad D_t\eta' = v'; \quad D_t\zeta' = w', \tag{15}$$

where $D_t = \partial/\partial t + \bar{u}\,\partial/\partial x$ is the rate of change following the mean zonal flow. The displacement vector $(\xi', \eta', \zeta')$ thus gives the location of a fluid parcel relative to the position which the same parcel would have had in the absence of the wave motion $(u', v', w')$. Corresponding to this displacement field is a generalized Lagrangian mean $(\bar{\ })^{\mathrm{L}}$ (Andrews & McIntyre 1978) which differs from the Eulerian mean $(\bar{\ })$ in that $(\bar{\ })^{\mathrm{L}}$ is the average along the wavy material line defined by the particle displacement field. Thus, as pointed out by Matsuno & Nakamura (1979), the Lagrangian mean motion is the motion of the centre of mass of a wavy material line which would be parallel to the $x$-axis in the absence of the wave motion. For small amplitude waves the Lagrangian mean can be related to the Eulerian mean by the Taylor series expansion

$$\bar{A}^{\mathrm{L}} = \bar{A} + \overline{\xi'\frac{\partial A'}{\partial x}} + \overline{\eta'\frac{\partial A'}{\partial y}} + \overline{\zeta'\frac{\partial A'}{\partial z}} + O(a). \tag{16}$$

For quasi-nondivergent motion the term $\overline{\zeta'\partial A'}/\partial z$ is $O(\epsilon)$ and from (9) and (15) it is clear that $\xi_x + \eta_y \approx O(\epsilon)$. Thus, correct to $O(\epsilon)$,

$$\bar{A}^{\mathrm{L}} = \bar{A} + \frac{\partial(\overline{\eta'A'})}{\partial y} = \bar{A} + \bar{A}_{\mathrm{S}}. \tag{17}$$

Here $\bar{A}_{\mathrm{S}} = \partial(\overline{\eta'A'})/\partial y$ is the *Stokes correction* which relates the Eulerian mean to the Lagrangian mean. This Lagrangian mean formalism provides a powerful tool for analysing wave transports.

The eddy flux $\overline{v'\chi'}$ can now be written in terms of wave transience and dissipation by multiplying (14) through by $\eta'$, averaging zonally and using the fact that $\overline{\chi' D_t \eta'} = \overline{\chi' v'}$ to obtain

$$\overline{v'\chi'} = -\overline{\eta'S'} + \frac{\partial}{\partial t}(\overline{\chi'\eta'}) + \overline{v'\eta'}\,\frac{\partial\bar{\chi}}{\partial y} + \overline{\eta'w'}\,\frac{\mathrm{d}X_0}{\mathrm{d}z}. \tag{18}$$

Defining a Lagrangian perturbation $\chi^{\mathrm{l}}$ by

$$\chi^{\mathrm{l}} \equiv \chi' + \eta'\bar{\chi}_y + \zeta' X_{0_z}, \tag{19}$$

where, from (14) $\quad D_t\chi^{\mathrm{l}} = D_t\chi' + v'\bar{\chi}_y + w'X_{0_z} = S',$

correct to $O(a^2)$, and noting that

$$\overline{v'\eta'} = \overline{\eta' D_t \eta'} = \tfrac{1}{2}(\overline{\eta'^2})_t,$$

it is possible to rewrite (18) in the form

$$\overline{v'\chi'} = -\overline{\eta'S'} + \frac{\partial}{\partial t}\left[\overline{\eta'\chi^{\mathrm{l}}} - \tfrac{1}{2}\overline{\eta'^2}\,\frac{\partial\bar{\chi}}{\partial y} - \overline{\eta'\zeta'}\,\frac{\mathrm{d}X_0}{\mathrm{d}z}\right] + \overline{\eta'w'}\,\frac{\mathrm{d}X_0}{\mathrm{d}z} + O(a^2). \tag{20}$$

Substituting from (20) into (13), and using the definition of the Stokes correction (17), then yields the zonal mean budget equation:

$$\frac{\partial\bar{\chi}}{\partial t} = -\bar{w}^{\mathrm{L}}\,\frac{\mathrm{d}X_0}{\mathrm{d}z} + \bar{S}^{\mathrm{L}} - \frac{\partial^2}{\partial t\,\partial y}\left[\overline{\eta'\chi^{\mathrm{l}}} - \tfrac{1}{2}\overline{\eta'^2}\,\frac{\partial\bar{\chi}}{\partial y} - \overline{\eta'\zeta'}\,\frac{\mathrm{d}X_0}{\mathrm{d}z}\right]. \tag{21}$$

In the absence of transience and damping (20) shows that

$$\overline{v'\chi'} = \overline{\eta'w'}\,\mathrm{d}X_0/\mathrm{d}z;$$

thus, the *horizontal* eddy flux is proportional to the *vertical* gradient of the mean concentration not to the horizontal gradient (Clark & Rogers 1979). Furthermore, the eddy flux convergence is in this case simply equal to the advection by the vertical Stokes drift $\overline{w}_S = \partial(\overline{\eta' w'})/\partial y$ (Wallace 1978). However, if transience and damping vanish, the non-interaction theorem of Andrews & McIntyre (1976*a*, 1978) requires that $\overline{w}^L = 0$ (i.e. that fluid parcels not change their mean heights). Thus from (21) there can be no net change in the mean concentration $\overline{\chi}$. In the long term mean, on the other hand, (21) implies a balance of the form

$$\overline{w}^L \mathrm{d}X_0/\mathrm{d}z = \overline{S}^L. \tag{22}$$

Thus, if $\mathrm{d}X_0/\mathrm{d}z > 0$, the centre of mass for a material line must gradually drift upwards in a region where the Lagrangian mean source, $\overline{S}^L$, is positive. For the case of potential temperature (22) has been used by Dunkerton (1978) to estimate the Lagrangian mean flow in the stratosphere and mesosphere based on computed heating rates for solstice conditions (with the approximation $\overline{S}^L \approx \overline{S}$).

Another interesting approximate form of the wave transport equation (21) can be obtained for transport of a conservative tracer ($\overline{S}^L = \chi^1 = 0$) by planetary waves. Now, $\eta'$ and $\zeta'$ are nearly in phase quadrature for vertically propagating planetary waves. Thus, $\eta'\zeta' \approx 0$ for these disturbances. If the rate of lateral particle dispersion is approximated using the Taylor (1915) eddy diffusion hypothesis,

$$K_{yy} = \tfrac{1}{2}\partial/\partial t(\overline{\eta'^2}),$$

then (21) can be written approximately as

$$\frac{\partial\overline{\chi}}{\partial t} = -\overline{w}^L\frac{\mathrm{d}X_0}{\mathrm{d}z} + \frac{\partial}{\partial y}\left[K_{yy}\frac{\partial\overline{\chi}}{\partial y}\right]. \tag{23}$$

Thus, the wave transport of a strongly vertically stratified tracer consists of the vertical motion of the centre of mass plus a horizontal 'diffusion' due to the dispersive effects of wave transience. It is important to realize, however, that both the Lagrangian mean vertical advection and horizontal diffusion are wave driven transports which in this case depend on wave transience.

In summary, it should now be clear that calculation of Eulerian eddy fluxes does not in itself give much insight into the net transport of a vertically stratified tracer. The generalized Lagrangian mean formalism of Andrews & McIntyre provides a much better framework for analysis of wave transport processes.

## 3. Equatorial waves

A particularly fascinating example of wave–mean flow interaction is provided by the interaction of the equatorially trapped Kelvin and mixed Rossby–gravity waves with the mean zonal wind in the tropical stratosphere to produce the quasi-biennial oscillation (Holton & Lindzen 1972). Both Kelvin and mixed Rossby–gravity waves are zonally propagating waves which are excited by forcing in the equatorial troposphere. The nature of this forcing is not known in any detail. However, latent heat release in convective storms appears to be the most likely source for the waves. These waves propagate vertically with comparatively short vertical wavelengths, and interact with the mean flow primarily through the mechanism of wave dissipation. As a result of this interaction process the Kelvin waves produce westerly accelerations of the mean zonal wind while the mixed Rossby–gravity waves produce easterly accelerations.

### (a) *Kelvin waves*

The Kelvin wave is an eastward propagating mode with negligible meridional velocity perturbations, and with zonal velocity and pressure perturbations which are in geostrophic balance and are symmetric with respect to the equator with latitudinal dependence given by

$$\exp\left[-\beta y^2/2\hat{c}\right],$$

where $y$ is the distance from the equator, $\beta \equiv 2\Omega/a$ is the value of $\mathrm{d}f/\mathrm{d}y$ at the equator, and $\hat{c}$ is the phase speed relative to the mean flow. For small dissipation rates the vertical wavelength of the Kelvin wave, $L_z \approx 2\pi\hat{c}/N$, where $N$ is the buoyancy frequency, depends only on the Doppler shifted phase speed $\hat{c}$, not on the zonal wavenumber and frequency separately. (An extensive discussion of the properties of Kelvin waves and other equatorial modes is given in Holton (1975).)

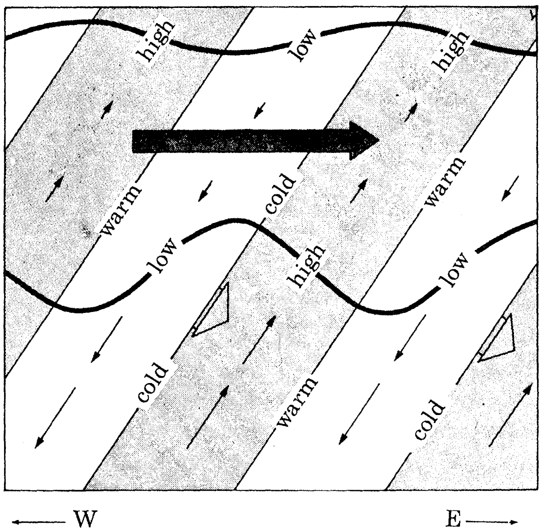


FIGURE 1. Longitude–height section along the equator showing pressure, temperature, and wind perturbations for a thermally damped Kelvin wave. Heavy wavy lines indicate material lines, short blunt arrows show phase propagation. Areas of high pressure are shaded. Length of the small thin arrows is proportional to the wave amplitude, which decreases with height owing to damping. The large shaded arrow indicates the net mean flow acceleration due to the radiation stress divergence.

The observed Kelvin waves in the stratosphere seem to be limited to zonal wavenumbers 1 and 2. The waves have periods in the range of 10–20 days and vertical wavelengths of 6–12 km (Wallace 1973). A longitude–height cross section showing the structure of a Kelvin wave in the presence of weak thermal dissipation is shown in figure 1. The phase relations among the zonal velocity, vertical velocity, pressure, and temperature perturbations are just those of an eastward propagating internal gravity mode. In particular, phase lines propagate downward, consistent with an upward energy propagation.

The role of thermal dissipation in wave–mean flow interaction can be elucidated in a particularly simple manner for the Kelvin wave case. In the absence of mechanical dissipation and horizontal shear of the mean flow the mean zonal flow acceleration due to Kelvin waves is given by

$$\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\,\overline{u'w'}\right). \tag{24}$$

Eliassen & Palm (1961) showed that for waves of steady amplitude in the absence of dissipation and critical levels (levels where the mean zonal wind speed equals the wave phase speed), $\rho_0\overline{u'w'}$ must be independent of height so that the mean flow forcing vanishes. To examine the role of thermal dissipation it is again convenient to introduce the linearized particle displacement $\zeta'(x, z, t)$, which satisfies $D_t\zeta' = w'$, where for steady amplitude waves with zonal phase speed $c$ the operator $D_t = -(c-\bar{u})\,\partial/\partial x$. The perturbation zonal momentum equation

$$D_t u' = -(1/\rho_0)\,\partial p'/\partial x,$$

thus implies that $\rho_0 u' = p'/(c-\bar{u})$. Therefore, the vertical momentum flux may be written

$$\rho_0\overline{u'w'} = -\overline{p'\,\partial\zeta'/\partial x},$$

so that the mean flow forcing is just

$$\frac{\partial\bar{u}}{\partial t} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left[-\overline{p'\,\frac{\partial\zeta'}{\partial x}}\right]. \tag{25}$$

The wavy material lines $\zeta'(x, t)$ are shown at two different levels by the heavy solid lines in figure 1. It should be noted that for adiabatic waves

$$D_t\theta' + w'\bar{\theta}_z = 0,$$

where $\theta$ is potential temperature. Thus $\zeta' = -\theta'/\bar{\theta}_z$ so that the material lines are also lines of constant potential temperature. However, in the thermally damped case depicted in figure 1, the potential temperature isolines should actually be shifted slightly eastward relative to the material lines since thermal damping destroys the quadrature relationship between $\theta'$ and $w'$. (No attempt has been made to portray this slight phase shift in the figure.)

Mean flow forcing by the Kelvin waves can be given a particularly simple physical interpretation by using the mean flow equation (25). Now, $-\overline{p'\zeta'_x}$ simply represents the net eastward stress due to the pressure force exerted by the fluid below a wavy material line on the fluid above the material line. In the case shown in figure 1 this so called 'radiation stress' is positive since $p' > 0$ where $\zeta'_x < 0$, and decreases with height because the waves are damped. Thus, the fluid contained between the two wave material lines is subject to a net eastward force due to the action of the radiation stress. The magnitude of this net force is, as indicated in (25), proportional to the rate of decrease of the radiation stress with height, which in turn depends on the rate constant for the damping. Observations indicate that the Kelvin waves in the equatorial stratosphere do have sufficient amplitude to drive the westerly phase of the quasi-biennial oscillation through forcing due to wave dissipation. Kelvin waves damped thermally are also the only plausible mechanism yet suggested for generating the westerly phase of the semiannual zonal wind oscillation in the equatorial upper stratosphere and mesosphere (Dunkerton 1979).

### (b) *Mixed Rossby–gravity waves*

The mixed Rossby–gravity mode is a westward propagating wave which has meridional wind perturbations symmetric about the Equator and zonal velocity and pressure perturbations antisymmetric with respect to the Equator. The latitudinal dependence of the pressure and zonal velocity perturbations in the mixed Rossby gravity wave turns out to have the form (Holton 1975)

$$y\,\exp\left[-\frac{(1-k\omega/\beta)\,\beta^2y^2}{2\omega^2}\right],$$

where $k$ and $\omega$ are the zonal wavenumber and frequency, respectively. These must satisfy the condition $0 < k\omega < \beta$ in order that valid solutions exist. Observed mixed Rossby–gravity waves have periods in the range of 3–5 days, zonal wavelength 10000 km and vertical wavelengths in the range of 4–8 km. These waves have their greatest amplitudes in the lower stratosphere during the westerly phase of the quasi-biennial oscillation. A longitude–height cross section showing the structure of a mixed Rossby–gravity wave subject to weak thermal dissipation is shown in figure 2. The diagram is for a latitude north of the equator. As in the corresponding Kelvin wave figure, wavy material lines are again denoted by heavy solid lines.

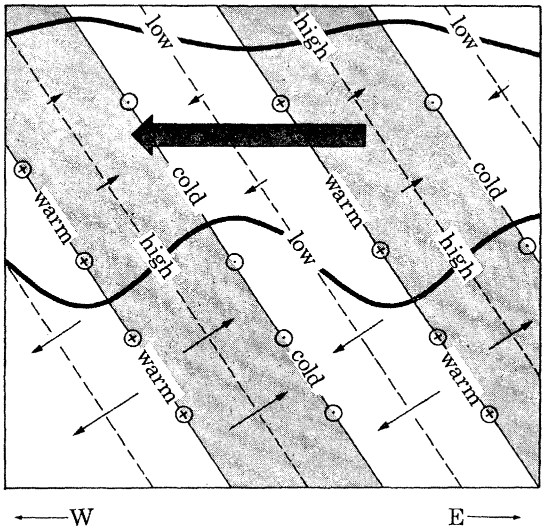


FIGURE 2. Longitude–height section along a latitude circle north of the equator showing pressure, temperature, and wind perturbations for a thermally damped mixed Rossby–gravity wave. Areas of high pressure are shaded. Small arrows indicate zonal and vertical wind perturbations with length proportional to the wave amplitude. Meridional wind perturbations are shown by arrows pointed into the page (northward) and out of the page (southward). The large shaded arrow indicates the net mean flow acceleration due to the radiation stress divergence.

Again the radiation stress concept may be used to elucidate the wave-mean flow interaction process. Andrews & McIntyre (1976*a*) have shown that for small amplitude equatorial waves that have vertical scales small compared to the vertical scale of the mean flow, the mean flow acceleration to lowest order is given by

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial}{\partial y} S_{xy} - \frac{\partial}{\partial z} S_{xz}, \tag{26}$$

where

$$S_{xy} = \overline{u'v'} - \bar{u}_z \overline{v'\Phi'_z}/N^2,$$

$$S_{xz} = \overline{u'w'} + (\bar{u}_y - f)\, \overline{v'\Phi'_z}/N^2.$$

Here $\Phi' = p'/\rho_0$, and the Boussinesq approximation has been used for simplicity. However, Eliassen & Palm (1961) showed that for steady inviscid waves

$$S_{xy} = \overline{v'\Phi'}/(c-\bar{u}), \quad S_{xz} = \overline{w'\Phi'}/(c-\bar{u}). \tag{27}$$

Again, introducing the particle displacements $\zeta'(x, y, z, t)$ and $\eta'(x, y, z, t)$, and observing that for steady waves

$$\zeta'_x = -w'/(c-\bar{u}); \quad \eta'_x = -v'/(c-\bar{u}),$$

the relationships of (27) can be rewritten in the form

$$S_{xy} = -\overline{\eta'_x \Phi'}; \quad S_{xz} = -\overline{\zeta'_x \Phi'}.$$

But, in the absence of mechanical dissipation, $\eta'_x$ and $\Phi'$ are in phase quadrature for mixed Rossby–gravity waves. Thus $S_{xy} = 0$ and (26) becomes

$$\partial \bar{u}/\partial t = +\partial(\overline{\Phi' \zeta'_x})/\partial z, \tag{28}$$

which is identical to (25) when the Boussinesq approximation ($\rho_0$ = constant) is applied. Thus, just as in the case of the Kelvin waves the mean flow forcing for mixed Rossby–gravity waves subject only to weak thermal damping can be expressed in terms of the vertical derivative of the radiation stress $-\overline{p'\zeta'_x}$. In the present case, however, it is clear from figure 2 that $p' > 0$ where $\zeta'_x > 0$ so that the radiation stress exerts a net westward force on the fluid contained between the two wavy material lines shown in the figure. Thus, the mixed Rossby–gravity waves interact with the mean flow to produce the easterly phase of the quasibiennial oscillation.

### (c) *The influence of mechanical dissipation and horizontal mean wind shear*

In the above discussion it was assumed that wave dissipation was due to thermal damping only. Addition of mechanical dissipation causes large changes in the wave–mean flow interaction process, especially in the case of the mixed Rossby–gravity wave. Since both the pressure and vertical displacement fields for a mixed Rossby–gravity wave are antisymmetric with respect to the equator, the mean flow acceleration, $\bar{u}_t$, for a thermally damped wave must, according to (28), be zero at the equator and have two easterly maxima symmetrically distributed on either side of the equator. However, as first shown by Andrews & McIntyre (1976*a*), addition of even a small mechanical dissipation drastically changes the distribution of $\bar{u}_t$. There are two reasons for this sensitivity. In the first place, mechanical dissipation introduces a non-zero viscous contribution to $S_{xy}$, which for Rayleigh friction dissipation has the form $-\lambda\overline{\eta' u'}$, where $\lambda$ is the rate coefficient for the damping. Secondly, mechanical dissipation destroys the quadrature relationship between $\eta'_x$ and $\Phi'$ so that $S_{xy}$ can no longer be neglected in (26). In fact, mechanical dissipation produces a strong divergence of the lateral radiation stress, $-\overline{\eta'_x \Phi'}$, at the equator so that when mechanical dissipation is comparable in magnitude to thermal dissipation, $\bar{u}_t$ has a strong maximum in amplitude at the equator. This is just the distribution of $\bar{u}_t$ that is observed in the quasi-biennial oscillation.

Horizontal mean wind shear also has strong effects on the distribution of $\bar{u}_t$. Calculations by Andrews & McIntyre (1976*b*) and by Simmons (1978) have shown that when only thermal dissipation is present there is a tendency for the mean flow acceleration pattern to enhance an existing cross equatorial horizontal shear. However, Boyd (1978) has shown that when mechanical dissipation is included the mean flow acceleration pattern tends to diminish cross equatorial shear. This tendency for the waves to symmetrize the mean flow with respect to the equator has also been demonstrated in a numerical model by Holton (1979).

One striking aspect of the above studies is the finding that the structure of the Kelvin wave is determined almost entirely by the mean zonal wind at the equator. Even a rather strong cross equatorial shear has very little influence on the zonal wind and pressure perturbations. This may be demonstrated by comparison of figures 3 and 4, which show the Kelvin wave structures in two numerical simulations, one with mean zonal flow symmetric about the equator and the other with strong linear shear across the equator. The principal difference

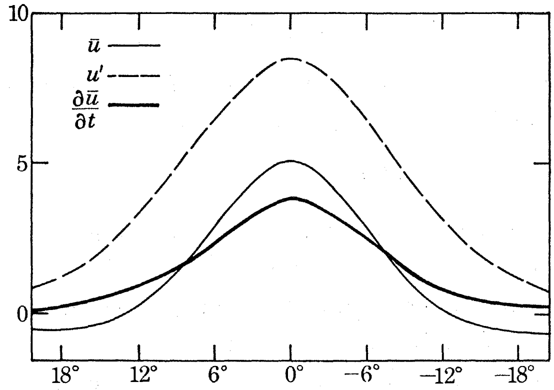


FIGURE 3. Perturbation zonal wind amplitude (m $s^{-1}$) and mean wind acceleration ($10^{-6}$ m $s^{-2}$) for a mechanically and thermally dissipated Kelvin wave in the presence of an equatorially symmetric mean flow.

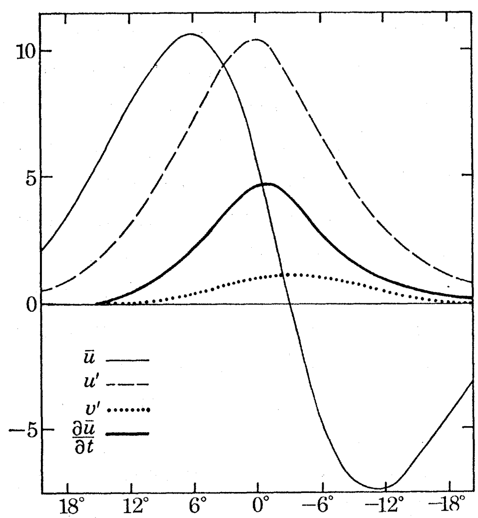


FIGURE 4. Perturbation zonal wind amplitude (m $s^{-1}$) and mean wind acceleration ($10^{-6}$ m $s^{-2}$) for a mechanically and thermally dissipated Kelvin wave in the presence of a mean flow with strong cross equatorial shear.

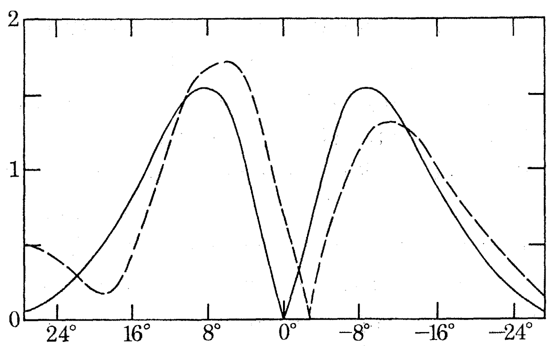


FIGURE 5. Perturbation zonal wind amplitude (m $s^{-1}$) for a mixed Rossby–gravity wave in the presence of a uniform mean flow (solid line) and a mean flow with linear shear of $-0.5$ (m $s^{-1}$)/100 km (dashed line).

between these two cases is that a weak asymmetric meridional wind perturbation is present in the latter case, and the maximum mean flow acceleration is shifted slightly into the hemisphere with the larger mean momentum deficit.

The structure of the mixed Rossby–gravity wave is more significantly affected by cross equatorial shear. The amplitudes of the zonal wind perturbation for two simulations, one with no mean wind shear, and the other with a weak horizontal shear $\bar{u}_y = -0.5\ (\mathrm{m\ s^{-1}})/100\ \mathrm{km}$, are shown in figure 5. In the sheared case the zero line is shifted into the hemisphere where the Doppler shifted frequency is largest while the peak amplitude is depressed in that hemisphere and enhanced in the other hemisphere. With larger horizontal shears, more dramatic distortions occur in the wave structure. However, in all mixed Rossby–gravity wave forcing cases investigated by Holton (1979) the wave driven mean flow acceleration acted to reduce the initial cross equatorial horizontal shear, and the mean flow eventually evolved as an easterly jet centred on the equator. Thus, it appears that as long as there is substantial mechanical damping as well as thermal damping, the equatorial wave–mean flow interaction process will tend to produce mean flow changes which are symmetric about the equator despite the asymmetric horizontal shears introduced by the annual cycle. This tendency may explain the remarkable equatorial symmetry observed for the quasi-biennial oscillation.

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